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ON THE THEORY OF SELF-EXCITING GYROSCOPES

[Following is a translation of an article by Doctor of Physicomathematical Sciences D. R. Merkin of the Leningrad Order of Lenin Forestry Engineering Academy published in Izvestiya Vysshikh Uchebnykh Zavedeniy (News of Higher Educational Institutions), Priborostroyeniye (Instrument Building), No. 3, 1959, pages 21-24.]

This article contains a discussion of the rotating motion of a symmetrical gyroscope located in a resistint medium, and acting under an arbitrary but assigned moment of self-excitation.

The problem of determining the angular velocity of a symmetrical gyroscope without taking into account torque an without analyzing the nature of its motion was investigated by U. T. Boedewadt [3] for the particular case when the moment of self-excitation maintains a constant magnitude and a constant direction relative to the body. R. Grammel [4, 5] investigated the stability of motion of a self-exciting symmetrical gyroscope, taking into account torque. The effect of torques without taking into account other forces was investigated by many authors; for example, see references [1, 3 and 6.]

Determination of Angular Velocity

Let a moment be produced in a symmetrical giroscope as a result of self-excitation, the torque changing according to an assigned rule. Then the Euler equations of motion, taking into account the torque, will be in the following form:

$$\begin{aligned} A\dot{p}+(C-A)rq &= -\varepsilon p+m_x(t); \\ A\dot{q}+(A-C)pr &= -\varepsilon q+m_y(t); \\ C\dot{r} &= -\varepsilon_1 r+m_z(t); \end{aligned} \qquad (1.1)$$

where p, q and r are the projections of angular velocity of the gyroscope along the axes x, y and z which are rigidly connected with the body; A and C are the moments of inertia; and ε and ε_1 are positive constants characteri ing the the torque (generally $\varepsilon \neq \varepsilon_1$); and $m_x(t)$, $m_y(t)$ and $m_z(t)$ are projections of the moment of self-excitation (assigned functions of time).

We find r(t) from the third equation.

$$r(t) = \gamma(t) e^{-\frac{\varepsilon_1}{C} t}, \quad \gamma(t) = r_0 + \frac{1}{C} \int_0^t m_z(t) e^{\frac{\varepsilon_1}{C} t} dt. \tag{1.2}$$

We introduce the derived value for r(t) into the first two equations of the system (1.1); we multiply the second equation by $i = \sqrt{-1}$ and we superpose both equations. Then, introducing the complex values $w = p + iq$ and $m(t) = m_x(t) + im_y(t)$, we get

$$\dot{w} + \left[\frac{\varepsilon}{A} \quad i \varkappa r(t)\right] w = \frac{1}{A} m(t), \tag{1.3}$$

where the real number $\varkappa$ is determined by the equality

$$\varkappa = \frac{C - A}{A}. \tag{1.4}$$

Integrating the linear equation (1.3), we find w(t).

$$w(t) = \left\{ w_0 + \frac{1}{A} \int_0 m(t) \exp\left[\frac{\varepsilon}{A} t - i\nu(t)\right] dt \right\} \exp\left[-\frac{\varepsilon}{A} t + i\nu(t)\right],$$

$$\nu(t) = \varkappa \int_0^t r(t)\, dt. \tag{1.5}$$

Expanding the real and imaginary parts, we get p(t) and q(t).

$$p(t) = e^{-\frac{\varepsilon}{A} t} J(t) \cos\mu(t), \quad q(t) = e^{-\frac{\varepsilon}{A} t} J(t) \sin\mu(t),$$

$$\mu(t) = \nu(t) + \eta(t), \quad \eta(t) = \operatorname{arc\,tg} \frac{J_2(t)}{J_1(t)}.$$

$$J_1(t) = p_0 + \frac{1}{A} \int_0^t \left[m_x(t) \cos\nu(t) + m_y(t) \sin\nu(t)\right] e^{\frac{\varepsilon}{A} t} dt.$$

$$J_2(t) = q_0 - \frac{1}{A} \int_0^t \left[m_x(t) \sin\nu(t) - m_y(t) \cos\nu(t)\right] e^{\frac{\varepsilon}{A}} dt. \tag{1.6}$$

$$J(t) = \sqrt{J_1^2(t) + J_2^2(t)}.$$

The angle ϑ between the axis of the gyroscope z and the vector of angular velocity Ω is determined by the

equality

$$\operatorname{tg} \vartheta = \frac{\sqrt{p^2 + q^2}}{r}.$$

or, using (1.2) and (1.6)

$$\operatorname{tg} \vartheta = \frac{J(t)}{\gamma(t)} e^{-\left(\frac{\varepsilon}{A} - \frac{\varepsilon_1}{C}\right)t}. \tag{1.7}$$

Thus, the angular velocity of the gyroscope moving in a resisting medium, acting under an arbitrary but assigned moment of self-excitation, has been fully determined. It is to be noted that from the total solution for projections of angular velocity (1.5) or (1.6) it is easy to obtain all the particular cases investigated by Boedewadt in reference [3]; for this purpose, however, it is sufficient to set up $\varepsilon = \varepsilon_1 = 0$ and $m_1 = $ const. When $m_1 = 0$, we obtain cases investigated in references [1, 2 and 6]. In particular, the angle between the axis of the gyroscope and the vector of angular velocity when $m_1 = 0$ will be determined by the formula:

$$\operatorname{tg} \vartheta = \frac{\sqrt{p_0^2 + q_0^2}}{r_0} e^{-\left(\frac{\varepsilon}{A} - \frac{\varepsilon_1}{C}\right)t}$$

When $\varepsilon = \varepsilon$ and $C > A$ the vector of angular velocity comes close to the axis of the gyroscope z, and when $C < A$ the vector of angular velocity comes close to the equatorial plane of the gyroscope -- a result which is well known in the literature [1, 2, 6].

Investigation of Motion at a Constant Moment of Excitation

Let the moment of self-excitation maintain a constant magnitude and a constant direction relative to the body. In this case, owing to the symmetry of the gyroscope we can assume, without violating the generalization, that m = 0. Then, when $m_x = $ const. and $m_z = $ const, we shall find r(t), v(t) and w(t) from (1.2) and (1.5):

$$r(t) = \frac{m_z}{\varepsilon_1} + \left(r_0 - \frac{m_z}{\varepsilon_1}\right) e^{-\frac{\varepsilon_1}{C}t};$$

$$v(t) = \frac{\varkappa m_z}{\varepsilon_1} t - \frac{C}{\varepsilon_1} \varkappa \left(r_0 - \frac{m_z}{\varepsilon_1}\right) e^{-\frac{\varepsilon_1}{C}t}; \tag{2.1}$$

$$w(t) = \left\{w_0 + \frac{m_x}{A}\int_0^t exp\left[\frac{\varepsilon}{A}t - i\,v(t)\right]dt\right\} exp\left[-\frac{\varepsilon}{A}t + i\,v(t)\right].$$

At the outset, we shall investigate a case in which the initial velocities satisfy the following conditions:

$$r_0 = r^* = \frac{m_z}{\varepsilon_1}; \quad w_0 = w^* = \varepsilon_1 m_x \frac{\varepsilon\varepsilon_1 + i\varkappa A m_z}{\varepsilon^2\varepsilon_1^2 + \varkappa^2 A^2 m_z^2}. \tag{2.2}$$

In this case, the solution (2.1) assumes the form:

$$r(t) = r^* \quad w(t) = w^*$$

and consequently, the projections of the vector of angular velocity along the axes x, y and z are rigidly connected with the gyroscope, and will maintain throughout the period of motion constant values, equal to the initial

$$r^* = \frac{m_z}{\varepsilon_1}, \quad p^* = \frac{\varepsilon\varepsilon_1^2 m_x}{\varepsilon^2\varepsilon_1^2 + \varkappa^2 A^2 m_z^2}, \quad q^* = \frac{\varkappa\varepsilon_1 A m_x m_z}{\varepsilon^2\varepsilon_1^2 + \varkappa^2 A^2 m_z^2}.$$

It is clear that in this particular case the motion of the gyroscope will consist of a uniform rotation about a fixed axis which coincides with the initial direction of the vector of angular velocity $\bar{\Omega}^*(p^*, q^*, r^*)$.

We shall now investigate a general case, when at the start of its motion the gyroscope has been endowed with an arbitrary initial velocity $\bar{\Omega}_0$ (p_0, q_0, r_0), which does not coincide with the initial velocity considered above, $\Omega^*(p^*, q^*, r^*)$. It follows from (2.1) that at arbitrary initial values of r_0 and $w_0 = p_0 + iq_0$, the magnitudes r(t) and w(t) have a tendency, at t → infinity, toward r* and w*, respectively. This means that at any initial conditions the gyroscope, acting under a constant moment of excitation and maintaining an unchanged direction relative to the body, asymptotically tends toward a uniform rotation about a fixed axis. The magnitude of the limit of angular velocity equals

$$\Omega^* = \sqrt{\frac{\varepsilon_1^2 m_x^2}{\varepsilon^2\varepsilon_1^2 + \varkappa^2 A^2 m_z^2} + \frac{m_z^2}{\varepsilon_1^2}}$$

The direction of the limit axis of rotation in a basic (fixed) system of calculation is easy to construe by the following method. The position of the moving system of coordinates x, y, z at the initial instant of time is fully determined by the initial position of the gyroscope. According to the determinable values of projections of angular velocity p* q* r* we shall set up vector $\bar{\Omega}^*$ in this initial position. Its direction relative to the basic system of calculation determines the direction of the limit axis of rotation of the gyroscope.

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